An Experimental Study on Long Transient Oscillations in Cooperative CNN Rings

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Abstract—The paper considers a class of one-dimensional circular standard cellular neural network (CNN) arrays with a typical three-segment piecewise linear activation and two-sided cooperative (positive) interactions (a cooperative CNN ring). Numerical simulations show that in a wide range of interconnection parameters, and for a wide set of initial conditions, the solutions of a cooperative CNN ring display unexpectedly long oscillations, lasting even hundreds of cycles, before they eventually converge toward an equilibrium point. The goal of this paper is to confirm the presence of such long-transient oscillations through laboratory experiments on a simple discrete-component prototype of a cooperative CNN ring with 16 cells and to analyze some of their salient features. Analytical results are also provided to support the numerical and experimental findings.

I. INTRODUCTION

In this paper we investigate some fundamental dynamical properties of a class of one-dimensional circular standard cellular neural network (CNN) arrays with a typical three-segment piecewise linear (PL) activation and two-sided interactions (a CNN ring).

We consider the important case where there are nonsymmetric cooperative (positive) interactions between neighboring neurons. In the cooperative case a CNN ring generates a monotone semiflow but, as pointed out in [1], due to the squashing effect of the horizontal segments (saturations) in the PL activations, the semiflow is not eventually strongly monotone (ESM). Recent papers have shown that, although the semiflow of a CNN ring is not ESM, yet it enjoys a LIMIT SET DICHTOMY and convergence properties analogous to those of ESM semiflows [2]–[5]. One main result in [2], [4], [5] is that the generic solution of a CNN ring converges toward an asymptotically stable (AS) equilibrium point (EP) in the long-run behavior. Such results have been obtained by extending some tools and techniques valid for ESM semiflows [6] so that they can be applied to the monotone semiflows generated by cooperative standard CNN rings.

Simulations with MATLAB of a cooperative CNN ring show that, according to the previous results, there is indeed convergence of solutions toward EPs. However, simulations also show that when the number of neurons is greater than or equal to 6, there is a wide range of interconnection parameters, and a wide set of initial conditions, for which the solutions display unexpectedly long transient oscillations before they eventually converge to EPs. Sometimes, even hundreds of oscillation cycles are displayed before a solution settles to a steady state.

The goal of this paper is to confirm the presence of such long-transient oscillations by means of laboratory experiments on a simple discrete-component prototype of a cooperative CNN ring with 16 cells and to analyze some of their salient features. The experiments are aimed at demonstrating that such long-lasting transient oscillations are not a numerical artifact of the simulation but are instead displayed by actual CNN circuits, moreover they are physically robust, namely they continue to be present also under circuit parameter variations due e.g. to tolerances and other nonidealities. Analytical results are also provided for supporting the numerical and experimental findings.

Cooperative CNNs have been widely used for solving some low-level image processing tasks in real time, such as noise removal, line detection, image smoothing and global connectivity detection [7]–[9]. They have been used also for cooperative learning [10] and for implementing some kinds of self-organizing maps [11]. We refer the reader to the papers [12], [13] for the first relevant investigations on stability of cooperative CNNs with and without interconnection delays. CNN rings with two-sided interconnections between neighboring neurons have received a great deal of attention in the literature. Although they are defined by one of the simplest cloning templates, they display a non-trivial dynamics that can be useful for solving some processing tasks as masked object extraction [14], shadow projection [15], and geometric template design [9]. The reader is referred to the papers [16], [17], [18], and the Ph.D. Thesis [11], for an investigation of other fundamental dynamic aspects of CNN rings, such as the presence of two main modes of propagation of the information signal corresponding to local diffusion and global propagation. Finally, we refer the reader to [19], [20] for the analysis and application of other related classes of neural networks, characterized by circulant interconnection matrices due to periodic boundary conditions, in the field of pattern formation and associative memories.
II. COOPERATIVE CNN RINGS

Let us consider an inputless one-dimensional standard CNN array satisfying the system of differential equations
\[ \tau \dot{x}_i = -x_i + \alpha g(x_{i-1}) + \beta g(x_{i+1}) \]  
(1)
where \( \tau > 0 \) is the neuron time constant, \( x_i, i = 1, 2, \ldots, n \), are the neuron state variables and \( g \) is the typical three-segment PL neuron activation
\[ g(\rho) = \frac{1}{2}(\rho + 1) - \frac{1}{2}(\rho - 1). \]
Each neuron has two-sided interactions \( \alpha, \beta \) with its nearest neighboring neurons. Note that the neuron self-connection is 0. We suppose that there are periodic boundary conditions, i.e., (1) is a ring (circular) CNN array. Due to the periodic boundary conditions, all indexes are considered modulo \( n \). For example, we have \( x_0 = x_n \) and \( x_{n+1} = x_1 \).

The interconnection matrix of (1) is given by
\[
A = \begin{pmatrix}
0 & \beta & 0 & 0 & \cdots & \alpha \\
\alpha & 0 & \beta & 0 & \cdots & 0 \\
0 & \alpha & 0 & \beta & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha & 0 & \beta \\
\beta & \cdots & 0 & 0 & \alpha & 0
\end{pmatrix}.
\]
Note that \( A \) is circulant and irreducible.

We assume henceforth that there are positive interactions \( \alpha, \beta > 0 \), i.e., the CNN (1) is cooperative. It has been shown in [2] that the cooperative CNN (1) generates a monotone solution semiflow \( \Phi \) but, due to the squashing effect of the horizontal segments (saturations) in the PL activation \( g, \Phi \) is not ESM. As a consequence, convergence results for (1) cannot be obtained from the standard results on convergence of ESM semiflows [6]. This notwithstanding, recent papers have shown that, although the semiflow \( \Phi \) of (1) is not ESM, due to the peculiar interconnecting structure of (1), it enjoys a LIMIT SET DICHOtOMY and convergence properties analogous to those valid for ESM semiflow [2], [4], [5]. One main result is that for generic values of parameters \( \alpha, \beta \) and for almost all initial conditions, a solution of (1) converges toward an AS EP. Such a property is known in the mathematical and engineering literature as almost convergence (almost complete stability) [10].

Let us consider the interconnection parameter space \( \alpha, \beta > 0 \). Also suppose for simplicity that \( \alpha + \beta > 2 \). Consider the curve
\[
C = \{ (\alpha, \beta) : (\alpha + \alpha \beta - \beta^2 - 1)(\beta + \alpha \beta - \alpha^2 - 1) = 0 \},
\]
the region within \( C \) given by
\[
R_\phi = \{ (\alpha, \beta) : \alpha > \frac{\beta^2 + 1}{\beta + 1}; \beta > \frac{\alpha^2 + 1}{\alpha + 1} \}
\]
and the region outside \( C \) given by
\[
R_\sigma = \{ (\alpha, \beta) : \alpha + \beta > 2 \} \setminus R_\phi
\]
as shown in Fig. 1.

It can be proved that within \( R_\sigma \) (1) has only three EPs. Namely, the origin, which is an unstable saddle-type EP, and the two saturated EPs
\[
\xi^+ = (\alpha + \beta, \alpha + \beta, \ldots, \alpha + \beta)' , \quad \xi^- = -\xi^+
\]
which are AS (the prime means transpose). Crossing curve \( C \) from \( R_\sigma \) to \( R_\phi \) leads to the birth of several additional EPs of (1). For example, when \( n = 16 \), for parameters \( (\alpha, \beta) = (3.020, 2.520) \in R_\sigma \) it can be checked by a numerical program that (1) has 5187 EPs (458 of these EPs are AS).

Numerical simulations with MATLAB of the ideal CNN model (1) show that, if \( (\alpha, \beta) \in R_\phi \), then the generic solution quickly converges toward one of the many AS EPs of (1). On the other hand, simulations show that if \( (\alpha, \beta) \in R_\sigma \), and the CNN ring has at least \( n = 6 \) cells, there are wide sets of initial conditions for which the corresponding solutions of (1) display unexpectedly long transient oscillations, even with hundreds of cycles, before they eventually converge to one of the AS EPs \( \xi^+, \xi^- \).

The main goal of this paper is to investigate on a simple experimental prototype the actual presence of the long-transient oscillations for the CNN ring (1) and some of their salient features (see Sections III, IV).

From an analytical viewpoint, as discussed in [21], the long-lasting oscillations are due to the presence of metastable rotating waves whose degree of instability quickly decreases with the dimension \( n \) of the CNN ring. More precisely, at the points \( P_P = (3.5, 2.5) \in R_\sigma, P_Z = (3.5, 0.0) \in \partial R_\sigma \), \( P_N = (3.5, -1.5) \not\in R_\sigma \), mathematical investigations of the ideal CNN model (1) for \( n = 2m \) and a normalized time constant \( \tau = 1 \), lead to the existence of a metastable
rotating wave $\Gamma^*_m$, $m \geq 3$, $s = P, Z, N$ [21]. By using root
asymptotics results on a certain class of lacunary polynomials,
the Floquet multipliers $\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_{2m-1}$ of $\Gamma^*_m$
can be estimated with great precision. The main results of [21] are
that
$$\lambda^*_1 > \lambda^*_0 = 1 > |\lambda^*_2| \geq |\lambda^*_3| \geq \ldots \geq |\lambda^*_{2m-1}|$$
(2)
and, with $\kappa_P = 12.90$, $\kappa_Z = 2.05$, $\kappa_N = 1.41$ and some
constants $c_s, d_s > 0$ (all independent of $m$), $s = P, Z, N$,
$$\lambda^*_1 < 1 + \frac{c_s}{\kappa_{m-3}} \quad \text{and} \quad |\lambda^*_2| < \frac{d_s}{\kappa_{m-3}}.$$  (3)
Letting $m \to \infty$, inequalities (2), (3) imply exponentially fast
asymptotics both for the dominant and the trivial eigenvector
and for the waveform as well.

Our numerical simulations with MATLAB suggest that
existence and metastability of $\Gamma^*_m$ extend to the parameter
set
$$R_\sigma = \{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq 2 \} \setminus \text{closure } (R_\phi)$$
containing the points $P_P, P_Z, P_N$. Analytically, existence of
$\Gamma_m$ is equivalent to root–finding for an auxiliary real function
$F(\alpha, \beta, \cdot) : \mathbb{R} \to \mathbb{R}$ depending on the sign of $\beta$ and also on
the maximal number of neurons in the linear region $[-1, 1]$ (and
making case $m = 3$ somewhat exceptional).\footnote{Since Floquet
asymptotics of periodic orbits are invariant with respect
to time scaling, one can work with the normalized time constant $\tau = 1$.
In numerical simulations with MATLAB and also with C++, $\Gamma_m$ emerges
as the periodic orbit to which the trajectory starting from $x_1(0) = \ldots = x_{m-1}(0) = 1, x_m(0) = 0, x_{m+1}(0) = \ldots = x_{2m}(0) = 0$ converges.
$\footnote{For $m \geq 4$ and $\{(\alpha, \beta) \in \mathbb{R}^2 : \beta < \alpha \} \cap R_\sigma$, there are only two
possibilities according as
$$\frac{\partial \log(0) + 1}{\alpha \log(0) - \beta} < g_{m-1} \quad \text{with} \quad \beta = \frac{\alpha - \beta + 1}{\alpha - \beta - 1}$$
or not: the maximum number of neurons (belonging to $\Gamma_m$) in the linear
region $[-1, 1]$ is either two or four. (However, for parameters outside $R_\sigma$
more precisely, slightly above the critical line $\alpha + \beta = 1$), this number seems to be $2m$.) For parameters in $R_\sigma$, metastability of $\Gamma_m$ can be reduced to the
stability of a pair of linear or quadratic polynomials, respectively. All in
all, it is the piecewise linear choice of the neuron activation function $\tau$
in (1) that makes such a large simplification possible. On the other hand, by
numerical evidence, the general picture is robust under slight perturbations of
our neuron activation function in the maximum norm.
}

III. IMPLEMENTATION OF A CNN CELL

The circuit implementing the basic cell for the $i$-th neuron,
and the connections with neighboring neurons, is a slight
variant of that originally proposed by Chua and Yang in [24],
and is reported in Fig. 2. There are four stages. The first stage
is an inverting adder implementing the weighted sum of the
inputs to the $i$-th neuron, i.e., the voltages $V_\alpha = g(x_{i-1})$ and
$V_\beta = g(x_{i+1})$ coming from the outputs of the neighboring
neurons $i - 1$ and $i + 1$. Namely, we have
$$V_a = -\frac{R}{R_\alpha} V_\alpha - \frac{R}{R_\beta} V_\beta = -\alpha g(x_{i-1}) - \beta g(x_{i+1})$$
i.e., the dimensionless positive interaction parameters $\alpha, \beta$ are
obtained as
$$\alpha = \frac{R}{R_\alpha}, \quad \beta = \frac{R}{R_\beta}$$
In the actual circuit we have chosen $R = 560 \Omega$, so that the
design equations for $R_\alpha, R_\beta$ are given as
$$R_\alpha = \frac{R}{\alpha} = \frac{560 \Omega}{\alpha}, \quad R_\beta = \frac{R}{\beta} = \frac{560 \Omega}{\beta}.$$

The second stage is a voltage controlled current source as that in
the Appendix of [24]. Under the following constraint
$$\frac{R_2}{R_1} = \frac{R_4 + R_5}{R_3}$$
it can be shown that we have
$$I = -\frac{1}{1000} V_a = \frac{\alpha}{1000} g(x_{i-1}) + \frac{\beta}{1000} g(x_{i+1}).$$
In the actual circuit we have chosen, in accordance with (4),
$R_1 = 1.8 \, \text{k}\Omega, R_2 = 2.7 \, \text{k}\Omega, R_3 = 1.8 \, \text{k}\Omega, R_4 = 1.2 \, \text{k}\Omega$ and
$R_5 = 1.5 \, \text{k}\Omega$, yielding
$$I = -\frac{1}{1000} V_a = \frac{\alpha}{1000} g(x_{i-1}) + \frac{\beta}{1000} g(x_{i+1}).$$

The third stage implements the state $x_i$ of the $i$-th neuron,
which is given by the voltage $V_C$ at the capacitor $C_e$, and the
neuron time constant $\tau = R_\tau C_e$. By the Kirchoff current
law we obtain
$$I = I_C + I_R = C_x \dot{V}_C \quad \text{with} \quad \frac{V_C}{\dot{C}_e} = C_x \dot{x}_i + \frac{x_i}{R_x}.\quad \text{We have chosen } R_x = 1 \, \text{k}\Omega,\ C_x = 680 \, \text{nF}, \text{ hence we have}$$
$$680 \times 10^{-9} \dot{x}_i = -\frac{x_i}{1000} + \frac{\alpha}{1000} g(x_{i-1}) + \frac{\beta}{1000} g(x_{i+1})$$
which coincides with the equation (1) describing the dynamics of
the $i$-th neuron with a time constant $\tau = 6.8 \times 10^{-4} \, \text{sec}$. The
fourth stage implements the output-function $g(x_i)$ of
the $i$-th neuron and also produces an amplification ratio
$$\frac{R_6 + R_7}{R_6}.$$ The op amp is followed by a voltage divider
with ratio
$$\frac{R_9}{R_8 + R_9} = \frac{R_6}{R_6 + R_7}$$
so that we have $V_Q = g(x_i)$. In the circuit, the parameter
values are chosen as $R_6 = 1 \, \text{k}\Omega, R_7 = 18 \, \text{k}\Omega, R_8 = 18 \, \text{k}\Omega$ and $R_9 = 1 \, \text{k}\Omega$. Finally, a fourth op amp implementing a
voltage-follower with $V_o = V_Q = g(x_i)$ is used for decoupling
the voltage divider from the input stage of each connected
neuron.
The circuitry for initial conditions is realized by using one switch per cell between the second and the third stage. When the switch is in 'load' position it disconnects the third stage from the second one. Moreover it connects the third stage to a voltage generator providing the initial condition through the same switch. In 'run' position the switch connects the second and the third stage as shown in Fig. 2.

IV. EXPERIMENTAL RESULTS

We have built and experimentally tested three laboratory prototypes of a CNN ring (1) with \( n = 4, 8, 16 \) neurons, respectively. They have been implemented by means of discrete-component devices as resistors, capacitors and operational amplifiers (op amps), according to the design procedure in Section III. We used resistors with 5 % tolerances, capacitors with 10 % tolerances and op amps TL084. The supply voltage for the op amps was set to \( \pm 20 \) V. The switches are implemented with the integrated circuit MAX333 and each switch has a 130 \( \Omega \) series internal resistor.

In the experiments no oscillation was observed in the case \( n = 4 \) for \((\alpha, \beta) \in R_{\sigma}\). In the case \( n = 6 \) we already observed quite long-lasting transient oscillations when \((\alpha, \beta) \in R_{\sigma}\). These oscillations were much longer in time when \( n = 16 \).

In what follows we report a number of experimental findings for the laboratory prototype of a CNN ring with \( n = 16 \) neurons. First we have chosen parameters (see Fig. 1)

\[ (\alpha, \beta) = (1.7, 1.2) \in R_{\sigma}. \]

Figures 3(a)-(b) display the state voltage \( V_{C1} \), and the output voltage \( V_{o1} \), respectively, of neuron 1 as obtained by oscilloscope measurements. The ideal initial condition on the state variables was chosen as

\[ x_0' = 2.9 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}, \]

where each component is in Volts. The circuitry for setting initial conditions is (deliberately) quite imprecise so that the actual initial conditions are expected to be different from the ideal ones by several percents. It is seen that the state \( V_{C1} \) and the output \( V_{o1} \) display a long oscillation lasting more than 50 cycles before converging to a steady state. Other experiments with the same ideal initial condition led quite often to much longer transient oscillations, sometimes with even hundreds of cycles before converging (we do not show these oscillations since different cycles would be barely distinguishable on a figure reporting the whole oscillation).

Figure 4(a) displays a magnified version of the initial part of the \( V_{C1} \) and \( V_{o1} \) transients in Fig. 3(a)-(b). The waveform of

\[ V_i = g(x_i) \]

\[ V'_i \]
Fig. 4. (a) First part of the transient oscillation of $V_{C1}$ and output $V_{o1}$ shown in Fig. 3(a)-(b). (b) Oscillation as obtained with SPICE, and (c) oscillation as obtained with MATLAB simulations, for the same parameters $(\alpha, \beta) = (1.7, 1.2)$ and initial condition $x_0''$.

$V_{C1}$ corresponds to a symmetric waveform where the upper and lower plateaus have almost the same duration. Such an experimental waveform is seen to be quite in agreement with that obtained by a SPICE simulation of the actual CNN circuit (Fig. 4(b)), and that obtained with a MATLAB simulation of the ideal CNN model (1) for the same parameters and initial conditions (Fig. 4(c)).

In a second experiment, for the same parameters $(\alpha, \beta) = (1.7, 1.2) \in R_\sigma$, we have chosen quite a different ideal initial condition

$$x_0'' = 2.9(1 1 1 1 1 1 1 1 1 1 -1 -1 -1 -1 -1)^t.$$  

The measured state $V_{C1}$ and output $V_{o1}$ of neuron 1 are seen to display a long-transient oscillation with about 150 cycles before convergence. From the initial part of this transient, which is reported in Fig. 5, it can also be noted that the oscillation is no longer symmetric, indeed the upper plateau is longer than the lower plateau. Thus, it seems that the CNN approaches during the transient a different metastable rotating wave with respect to that in Fig. 4. Other simulations with other different sets of initial conditions have in fact shown the existence of several metastable waveforms with different lengths of the upper and lower plateaus.

In a third experiment we have considered different parameters $(\alpha, \beta) = (3.5, 2.5) \in R_\sigma$ with respect to the previous experiments (see Fig. 1). Figure 6 reports one of the obtained oscillations for $V_{C1}$, displaying more than seventy cycles before converging, with an ideal initial condition $x_0''$.

On the basis of the previous experiments, and several other experiments with different parameters $(\alpha, \beta) \in R_\sigma$ and different sets of initial conditions, we can conclude that the CNN ring with 16 cells frequently displays long oscillations, lasting even hundreds of cycles. Depending on the structure of the initial conditions there are oscillations that significantly differ in waveform and length of the upper and lower plateaus. We have also found that, for the same initial conditions, the oscillations in the real CNN circuit are in general shorter than those displayed by SPICE simulations and by the ideal CNN model (1). It is expected that the main reason for the shorter transients in the actual CNN circuit is the presence of tolerances and other biasing effects and offsets of the real op amps, which tend to destroy the circular symmetry of the CNN ring.

V. DISCUSSION AND CONCLUSION

The paper has pointed out that the solutions of CNN rings with two-sided cooperative interconnections can display long-lasting transient oscillations, before they eventually converge
to an EP, for a wide set of interconnection parameters and for wide sets of initial conditions. The presence of such oscillatory phenomena has been confirmed through experiments on a discrete-component laboratory prototype of a CNN ring. No sophisticated techniques were used to implement the circuit. Moreover, no fine tuning of the parameters was needed. The experimental results thus show that the oscillations are physically robust with respect to tolerances and other nonidealities in the electronic implementation.

Due to the presence of long-lasting transient oscillations there are difficulties to distinguish between the (very long) transient period and the steady state behavior of the solutions (convergence toward an EP). We can expect that the transient might be exceedingly long for practical use of CNN arrays in the solution of signal processing tasks in real-time.

It is believed that a deep theoretical analysis for explaining the basic phenomena leading to the presence of the long-transient oscillations is of crucial importance for better understanding the real-time processing capabilities of CNN arrays and neural network paradigms in general. Some abstract results have been already obtained in [21] where it is shown that the observed oscillations are due to the presence of metastable periodic solutions whose degree of instability is exponentially decreasing with the dimension \( n = 2m \) of the CNN ring.

ACKNOWLEDGEMENT

The Authors are indebted to Professor Tamas Roska who has closely followed the preparation of this paper and encouraged them to start research into this direction. The support of the grants TÁMOP-4.2.1.B-11/2/KMR-2011-0002 and TÁMOP-4.2.2/B-10/1-2010-0014 is gratefully acknowledged.

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