Chapter III.

Digital Filter Design (FIR & IIR)

Problems to be solved

• How to choose a suitable type of filters for the given specifications?
  How to justify your selection?

• How to design a digital IIR filter from an analog filter?
  (by manual calculation and by Matlab)

• How to design a digital linear-phase FIR filter?
  (by manual calculation and by Matlab)

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General consideration

Choose FIR or IIR filter?
Which type of FIR/IIR?

These aspects maybe important to your decision:
Control of stability
Computational cost
Linear-phase, or desired magnitude response

Constraint in filter design: phase and magnitude

Causality
Causal $\Rightarrow$ Physical realizable
e.g.: an ideal lowpass filter is non-causal

$$H(e^{j\omega}) = \begin{cases} 0 & |\omega| > \omega_p \\ \frac{h(\omega)}{2} & \omega_p < |\omega| < \pi \\ \text{otherwise} & \end{cases} \Rightarrow h(n) = \frac{\sin(\pi \omega_p n)}{\pi n}, \quad n \neq 0$$

Paley-Wiener theorem: Necessary and sufficient conditions for causal $h(n)$

If $h(n)$ has finite energy and $h(n)=0$ for $n<0$, then $\int_0^\pi |H(e^{j\omega})|^2 d\omega = \infty$
Conversely, if $|H(e^{j\omega})|$ is square integrable, and the integral is finite,
Then $|H(e^{j\omega})|$ can be associated with $\Theta(\omega)$, so that $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\Theta(\omega)}$ is causal.

Constraint in filter design: phase and magnitude

From the theorem

$\Rightarrow$ for a causal filter:

$|H(e^{j\omega})|$ cannot be zero at any finite band of frequencies, except some frequency points.

The transition from passband to stopband can not be infinitely sharp.

$H(\omega)$ and $H(\omega)$ are inter-dependent by the discrete Hilbert transform

$$H_{re}(\omega) = -\frac{1}{\pi} \int_0^{\pi} H(j) \cot(j) \frac{e^{j\omega L}}{2} dk$$

$\Rightarrow \Theta(\omega)$ and $|H(\omega)|$ cannot be arbitrarily chosen.
Digital IIR filter design

Filter specifications
Filter design: a causal and physically realizable LTI system
DE:
\[ y(n) = \sum_{k=1}^{N} a_k x(n-k) + \sum_{k=1}^{M} b_k x(n-k) \]

Task:
Select \( \{a, b\} \) which best approximate the desired \( H(e^{j\omega}) \)

Select criteria based on the magnitude / phase response of filter

e.g.: Frequency-selective characteristics based on magnitude response
- passband edge frequency \( \omega_p \)
- stopband edge frequency \( \omega_s \) (or transition region \( \omega_s - \omega_p \))
- maximum tolerable attenuation in the passband \( 2\delta_p \)
- minimum tolerable attenuation in the stopband \( 2\delta_s \)

Design of digital IIR filter

(a) Methods based on mapping analog filters to digital filters

Frequency mapping:
S plane: left-side \( \leftrightarrow \) Z plane: inside the unit circle

Basic design approaches: (map analog IIR filters to digital IIR filters)

Approach 1:
- \( \text{design analog filter}
\)\( \rightarrow \) \( \text{design IIR filter}
\)

Approach 2:
- \( \text{design analog filter prototype}
\)\( \rightarrow \) \( \text{design IIR filter}
\)
Method 1. Approximation of derivatives

Use the backward difference (T=1)

\[
\frac{dy(t)}{dt} \bigg|_{t_0} = \frac{y(n) - y(n-1)}{T}
\]

\[= T \frac{1-z^{-1}}{T} \]

Transformation (mapping):

\[ s = \frac{T}{1-z^{-1}} \]

Limitation:
Only suitable for lowpass and a limited bandpass filters

Method 2. Impulse invariance

\[ h(n) = h(t) \delta(t) \]

Transformation (mapping):

\[ z = e^{j\frac{\pi}{T}} \]

Using \( z = re^{j\theta} \), \( s = \sigma + j\Omega \) ⇒

\[ \sigma = e^{-\frac{\pi}{T}} \]

\[ \omega = \Omega \frac{T}{\pi} \]

T must be sufficiently small to avoid aliasing.

Limitations:
Only suitable for lowpass and a limited bandpass filters

Method 3. Bilinear transformation

\[
\frac{2}{1-z^{-1}} \frac{2}{1+z^{-1}}
\]

Using \( z = re^{j\omega} \), \( s = \sigma + j\Omega \)

⇒ digital-analog frequency warping:

\[
\Omega = \frac{2}{T} \tan \left( \frac{\omega}{2T} \right) \frac{\pi}{T}
\]

\[ \sigma = 2\tan \left( \frac{\omega}{2} \right) \frac{T}{\pi} \]

Where: digital frequency \( \omega = \frac{2\pi f}{T} \)

analog frequency \( \Omega \)

e.g.:
Analog frequency-band transformations

\( s \rightarrow s' \)

Prototype lowpass \( \rightarrow \) lowpass
\( \rightarrow \) bandpass
\( \rightarrow \) highpass

Analog frequency-band transformations

- Lowpass to lowpass transformation:
  given lowpass filter: \( \Omega \)
  passband edge frequency of the filter being transformed to \( \Omega' \)
  apply transformation: \( s \rightarrow \frac{\Omega'}{\Omega} s \)

- Lowpass to highpass transformation:
  passband edge frequency of the given lowpass filter \( \Omega \).
  passband cutoff frequency of the highpass filter \( \Omega' \).
  apply transformation: \( s \rightarrow \frac{\Omega'}{\Omega} s \)

- Lowpass to bandpass transformation:
  passband edge frequency of the given lowpass filter \( \Omega \).
  Edge frequencies of bandpass filter at:
  lower-passband \( \Omega_b \) and upper-passband \( \Omega_u \).
  Apply transformation: \( s \rightarrow \frac{s + j\Omega}{s + j\Omega_b} \)

- Lowpass to bandstop transformation:
  passband edge frequency of the given lowpass filter \( \Omega \).
  Edge frequencies of bandstop filter at:
  lower-passband \( \Omega_b \) and upper-passband \( \Omega_u \).
  Apply transformation: \( s \rightarrow \frac{s^2 + j\Omega}{s^2 + j\Omega_b} \)
**Digital frequency-band transformations**

\[ z \rightarrow z' \]

Prototype lowpass \(\rightarrow\) lowpass
\(\rightarrow\) bandpass
\(\rightarrow\) highpass

Table 7.2 (p.445 Mitra’s book)

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**Analog Butterworth (maximum flat) filters: review**

Design based on magnitude square

\[
|h(\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_p)^2} = \frac{1}{1 + (\Omega / \Omega_c)^2}
\]

- \(N\): filter order
- \(\Omega_p\): passband edge frequency in rad/sec.
- \(\Omega_c\): Passband edge frequency with 3db attenuation.

\(|h(\Omega)|\): flat both in passband and in stopband.

---

**Observations:**

- \(\Omega = 0\) : \(|h(j\Omega)| = 1\)
- \(|h(j\Omega)|\) is a monotonic decreasing function of \(\Omega\)
- \(|h(j\Omega)|\) approaches to an ideal lowpass filter when \(N \rightarrow \infty\)
- \(|h(j\Omega)|\) is maximally flat at \(\Omega = 0\)

(Derivatives of all orders exist and are equal to zero at \(\Omega = 0\)).
Determine the system function $H(s)$

- Substitute $\Omega = \frac{s}{j}$ to $|H(\Omega)|^2 \Rightarrow H(s)H(-s) = \frac{1}{|H(j\Omega)|^2}$

- Find poles of $H(s)H(-s)$

  $p_k = (\Omega^2)^{k} \Omega = \Omega e^{j \frac{2\pi k}{2N}} \quad k = 0, 1, ..., 2N-1$

  Equally distributed on a circle, radius $r = \Omega$, $\Delta\Omega = \frac{\pi}{N}$

  $p_k = \Omega e^{j \frac{2\pi k}{2N}} \quad N = e^{odd}$, $2k + N + 1 = 2m + 1$

  $N = even$, $2k + N + 1 = 2m + 1$

- Use stable and causal conditions, select poles of $H(s)H(-s)$ from the left-half of s-plane

  $\Rightarrow H_s(s) = \prod_{k=0}^{2N-1} \frac{1}{s - p_k}$

Poles of an analog Butterworth filter

- All poles are symmetrically located with respect to $j\Omega$ axis.

- Pole never falls on the imaginary axis;

- Pole falls on the real axis only when $N = odd$.

All poles are located on a circle ($r = \Omega$)

Steps for designing digital filter $H(z)$

Given filter specifications: $\omega_r$, $R$, and $\omega_c$, $A_c$.

1. Pre-warp the digital cutoff frequency $\omega_c$ and $\omega_r$, i.e., calculate the corresponding analog $\Omega_c$ and $\Omega_r$.

2. Design analog prototype filter $H(s)$ satisfying the specifications $\Omega_c$, $R$, and $\Omega_r$, $A_c$.

3. Frequency band transformation to obtain the actual analog frequency scale $s \rightarrow s'$

4. Use bilinear z-transform to obtain digital filter $H(z)$.

$H(z) = H(s) \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$
Examples:
Butterworth analog and digital IIR filter design

1. Convert an analog filter to a digital filter

2. Design a single pole digital lowpass filter with 3-dB bandwidth of 0.2π, using bilinear transformation to the analog filter

\[ H(s) = \frac{\Omega}{s + \Omega} \]

where \( \Omega \) is the passband edge frequency corresponds to 3-dB attenuation in the analog filter.

3. Design an analog filter from the given specifications

Analog Chebyshev filters
Chebyshev-I: equiripple response in the passband, Chebyshev-II: equiripple response in the stopband.

For Chebyshev-I filters:

\[ |H(j\omega)| = \frac{1}{1 + \epsilon T_n \left( \frac{\omega}{\Omega} \right)} \]

\( \epsilon \): passband ripple factor related to \( R_p \), \( T_n \): Nth-order Chebyshev polynomial

\[ T_n(s) = \begin{cases} \cos(n \cos^{-1}(s)) & 0 \leq s \leq 1 \\ \cosh(n \cosh^{-1}(s)) & 1 \leq s < \infty \end{cases} \]

where \( s = \frac{s}{\Omega} \)

Roots of analog Chebyshev filters
Distribution of roots of \( H(j\omega) \):

on an ellipse, with major axis \( \delta \Omega \), minor axis \( \delta \Omega \).

Design digital Chebyshev filters:

Similar method (as converting Butterworth analog to digital filter) can be applied to convert analog to digital Chebyshev filter.
Digital FIR filter design

Conditions for linear-phase

A FIR filter has a linear phase if the filter impulse response $h(n)$:

$h(n) = \pm h(M - 1 - n)$ for $n = 0, 1, \ldots, M - 1$ (symmetric/antisymmetric)

Or, $H(z) = \pm z^{-M-1} H(z^{-1})$

$\Rightarrow$ Roots of $H(z) = \text{roots of } H(z^{-1})$

$\Rightarrow$ Roots occur in pairs $\left\{ \frac{1}{z}, \frac{1}{z} \right\}$

- Roots occur in complex-conjugate pairs $\left\{ \frac{1}{z}, \frac{1}{z} \right\}$ for real $h(n)$

Linear-phase FIR

If $h(n)$ is symmetric $h(n) = h(M - 1 - n)$

$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta}$

$$
|H(e^{j\omega})| = e^{-j\frac{\pi}{2} \sum_{k=0}^{M-1} h(k) \sin \omega k} \\
\theta = \begin{cases} 
\frac{\pi}{2} \sum_{k=0}^{\frac{M-1}{2}} h(k) \cos \omega k & \text{if } M = \text{even} \\
\frac{\pi}{2} \sum_{k=0}^{\frac{M-1}{2}} h(k) \sin \omega k & \text{if } M = \text{odd}
\end{cases}
$$

If $h(n)$ is antisymmetric $h(n) = -h(M - 1 - n)$

$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta}$

$$
|H(e^{j\omega})| = e^{j\frac{\pi}{2} \sum_{k=0}^{M-1} h(k) \sin \omega k} \\
\theta = \begin{cases} 
-\frac{\pi}{2} \sum_{k=0}^{\frac{M-1}{2}} h(k) \cos \omega k & \text{if } M = \text{even} \\
\frac{\pi}{2} \sum_{k=0}^{\frac{M-1}{2}} h(k) \sin \omega k & \text{if } M = \text{odd}
\end{cases}
$$
Choose between symmetric/antisymmetric FIR

This depends on the applications

1. antisymmetric + M odd $\Rightarrow H(0) = H(\pi) = 0$
   not suitable for lowpass and highpass filters

2. antisymmetric + M even $\Rightarrow H(0) = 0$
   not suitable for lowpass filters.

3. symmetric + M odd, or, even $\Rightarrow H(0) \neq 0$
   suitable for lowpass filters

Problem formulation in FIR filter design

DE: $y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$
FIR filter: $H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$

Determine $H(z)$:
   according to the frequency specifications of $H(e^{j\omega})$

Window-based method

Given: $H_f(\omega)$

1. Calculate desired impulse response $h_d(\omega)$
   IDTFT: $h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$ (res(=0))

2. Obtain $h(n)$ by multiplying
   $h(n) = w(n)h_d(n) = \begin{cases} h_d(n) & n = 0, 1, \ldots, M - 1 \\ 0 & \text{otherwise} \end{cases}$
   $w(n)h_d(n) \leftrightarrow W(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega})$

Select window function $w(n)$ satisfying:
   main lobe width and attenuation
   transition bandwidth
   side lobe attenuation
Examples of FIR filter design

e.g.1: Design an FIR filter by windowed method.

The filter must satisfy the following specifications:
- passband edge frequency: 1.5kHz
- maximum transition bandwidth: 0.5kHz
- minimum stopband attenuation: 50dB
- sampling frequency: 8kHz

e.g.2: Select window function and determine the order of the FIR filter.

The filter has the following specifications:
- passband frequencies: [150 Hz, 250 Hz]
- maximum transition bandwidth: 50Hz
- maximum passband ripple: 0.1dB
- minimum stopband attenuation: 60dB
- sample frequency: 1kHz

Frequency sampling-based method

Given \( h(n) \)

uniformly sampling N points from the frequency response \( H(e^{j\omega}) \)

\[
H(e^{j\omega}) = H(e^{j2\pi \frac{k}{N}}) \Delta H(k) \quad k = 0,1,...,N-1
\]

Inverse DFT:

\[
h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j2\pi \frac{k}{N} n} = \sum_{m} h_{0}(n+mN) \quad 0 \leq n \leq N-1
\]

If \( h_{0}(m) \) is finite length of M, and \( M \leq N \Rightarrow h(n) = h_{0}(n) \)

e.g.

Computer aided-design of FIR filters

( criterion-based & iterated implementation )
Optimal linear-phase FIR filter design using Chebyshev approximation

For a linear-phase FIR filter \( H(z) = e^{-j\omega_0 z^{-1}} H(z) \) \( \beta = 0 \) or \( \pi/2 \)

\[
H(z) = \sum_{k=0}^{N} a(k) \cos(\omega_k)
\]

(Assume: \( H(z) \) is symmetric and odd)

Criterion: (minimax criterion)

minimizing the maximum weighted approximation error between the desired and the actual frequency response

\[
e\left(\frac{P(\omega)}{H(\omega)} - \frac{|H(\omega)|^2}{P(\omega)}\right) \sum_{k=0}^{\infty} a(k) \cos(\omega_k)
\]

determine \( \{a_i\} \) such that

\[
e = \min_{\epsilon} \left[ \max_{\omega \in S}\left|\epsilon(\omega)\right|\right]
\]

S: passband and stopband of the desired filter, i.e., \( S = [\omega_0, \pi] \)

Weighing function \( P(\omega) = \begin{cases} \delta_k/\delta_k & \text{passband} \\ \delta_k & \text{stopband} \end{cases} \) (\( \delta_k \) normalized)

**Alternation theorem** (by Parks and McClellan).

Let \( S \) be defined in \( (0, \pi) \)

A necessary and sufficient condition for \( H(\omega) = \sum_{k=0}^{\infty} a(k) \cos(\omega k) \) to be the unique, best weighted Chebyshev approximation to \( H(\omega) \) in \( S \), is that there exist at least \((L+2)\) extrema in \( S \), i.e., \( \{\omega_i\} \in S : \omega_0 < \omega_1 < \cdots < \omega_{L+2} \)

\[
e(\omega_i) = -e(\omega_{i+1})
\]

and \( |e(\omega_i)| = \max_{\omega \in S} |e(\omega)| \), \( i = 1, 2, \ldots, L+2 \)

This theorem guarantees a unique solution to the problem.

**Algorithm**

Set \((L+2)\) equations at the desired extremal frequencies \( \{\omega_i\} : i = 0, 1, \ldots, L+1 \)

\[
\frac{P(\omega_i)}{H(\omega_i)} - \frac{|H(\omega_i)|^2}{P(\omega_i)} = -e(\omega_i)
\]

\[
\Rightarrow H(\omega_i) + \frac{1}{P(\omega_i)} = \frac{|H(\omega_i)|^2}{P(\omega_i)}
\]

\[
\Rightarrow \sum_{k=0}^{\infty} a(k) \cos(\omega_i k) \left(\frac{1}{P(\omega_i)} - H(\omega_i)\right) = 0
\]

(1)

With unknown parameters: \( \{\omega_i, i = 0 \cdots L+1\} \)

\( \{e, a(k), k = 0, 1 \cdots L\} \)
**Algorithm**

**Remez iterative algorithm:**

1. initial guess of extremal frequencies \( \omega_i, i = 0, 1, \ldots, L+1 \);
2. determine \( H(k) \) and \( \epsilon \) in Eq.(1) (assume \( \epsilon_j \) are known);
3. Use \( \hat{\epsilon}(k) \) to interpolate \( H(a) = \frac{1}{L+1} \sum_{k=0}^{L+1} \hat{\epsilon}(k) e^{j\omega k} \);
4. compute new error: \( \epsilon(a) = |H(a) - e^{j\omega a}| \);
5. From \( \epsilon(a) \), select new \( L+2 \) largest extremal frequencies \( \{ \omega_i, i = 0, 1, \ldots, L+1 \} \);
6. repeat steps (2)-(5) until the algorithm converges.

**Filter design using Matlab**

IIR filters:
- `butter()`
- `cheby1()`
- `cheby2()`
- `ellip()`

FIR filters:
- `remez()`
- `fir1()`
- `fir2()` % windowed approach

Using `sptool` for a variety of choice and design

e.g.

**Group discussions**

1. Select one of the problems listed at the beginning of this chapter for the discussion.
2. Which concepts / methods do you still feel confused?